# Semidefinite programming hierarchies for polynomial programs

CMU 10-725 Optimization
Fall 2012

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(yes, this is going to be handwritten.)

There's life beyond convex optimization.

(Now you tell us...)

# Example 1: Sparse estimation

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Given: $\{(x_t, y_t)\}_{t=1}^{T}$ $\quad x_t \in \mathbb{R}^n$, $\; y_t = \langle w^*, x_t \rangle + \varepsilon_t$

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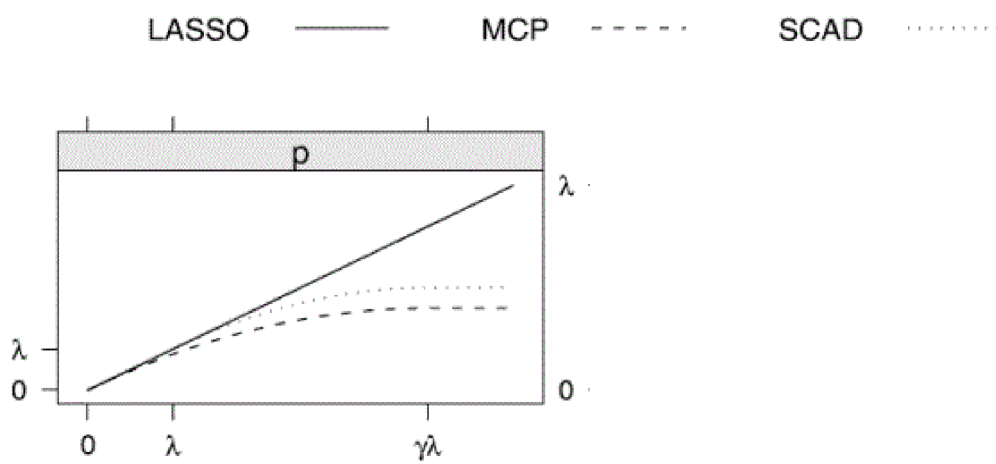
Statistically, not an ideal estimator. [Zha12]

- if $\min_j |w_j^*|$ is large, $\|w - w^*\|$ is suboptimal by a factor of $\sqrt{\ln(n)}$ ("Lasso bias")
- under "$\ell_2$ regularity conditions", $w$ does not have the correct support (even asymptotically)

MCP penalty: $\min_{w} \sum_{t=1}^{T} (y_t - \langle w, x_t \rangle)^2 + \lambda \sum_{j=1}^{n} \rho(|w_j|)$

$$\rho(|w_j|) = \begin{cases} \lambda |w_j| - w_j^2/2\gamma & |w_j| \le \lambda\gamma \\ \frac{1}{2}\gamma\lambda^2 & \text{otherwise} \end{cases}$$




[Bre11]

# Example 2: Linear Classification

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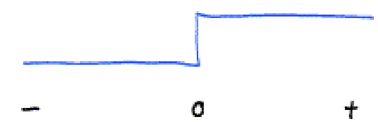
Given: $\{(x_t, y_t)\}_{t=1}^{T}$ $\quad x_t \in \mathbb{R}^n, \quad y_t \in \{-1, 1\}$ from distribution $D$

Goal: $\min\limits_{w} \mathop{\mathbb{E}}\limits_{(x,y)\sim D} \left[ \mathbb{1}(\langle w, x_t \rangle \neq y_t) \right]$

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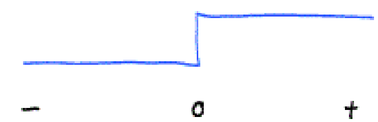
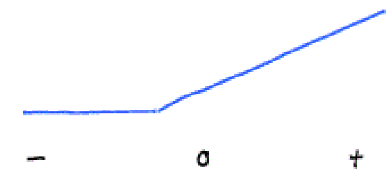



0-1 loss

SVM: $\min\limits_{w} \max(0, 1 - y_t \langle w, x_t \rangle) + \lambda \|w\|_2^2$




hinge loss

Relation to original problem is hazy

- need margin or exponential amount of data [Sha10]

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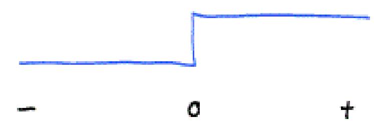
Given: $\{(x_t, y_t)\}_{t=1}^{T}$ $\quad x_t \in \mathbb{R}^n, \quad y_t \in \{-1, 1\}$ from distribution $\mathcal{D}$

Goal: $\min_{w} \; \mathbb{E}_{(x,y)\sim \mathcal{D}} \left[ \mathbb{1}(\langle w, x_t \rangle \neq y_t) \right]$



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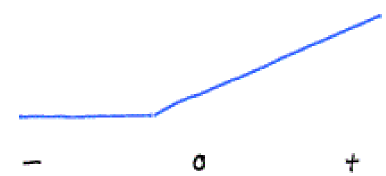

hinge loss

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Classical learning theory supports regularized ERM:

$$\min_{w} \; \sum_{t=1}^{T} \mathbb{1}\left[ \operatorname{sgn}(\langle w, x_t \rangle) \neq y_t \right] \quad \text{s.t.} \quad \|w\|_2^2 \leq B$$

# Polynomial optimization

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$$\min.\ p(x) \quad \text{s.t.} \quad g_1(x) \geq 0, \ldots, g_m(x) \geq 0$$

where $p$ and $g_i$ are polynomials

- in $n$ variables $x_1, \ldots, x_n$
- of degree $d$
- with rational coefficients

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- Very general.
- Not always well-defined.

# Classification as a polynomial program:

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$$\min_{w} \sum_{t=1}^{T} \ell_t \quad \text{s.t.} \quad B^2 - \sum_i w_i^2 \geq 0$$

$$\ell_t = \begin{cases} 1 & w \text{ incorrect on } x_t \\ 0 & \text{otherwise} \end{cases}$$

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$$\min_{w} \sum_{t=1}^{T} l_t \quad \text{s.t.} \quad B^2 - \sum_i w_i^2 \geq 0$$

$$\left[\, l_t = \begin{cases} 1 & w \text{ incorrect on } x_t \\ 0 & \text{otherwise} \end{cases}\right.$$

$$l_t \in \{0,1\} \longleftrightarrow l_t = l_t^2 \longleftrightarrow \begin{array}{l} l_t^2 - l_t \geq 0 \\ l_t - l_t^2 \geq 0 \end{array}$$

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$$\|w\|_2 \geq \varepsilon$$

## polynomials as vectors (in basis of monomials)

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$$= \langle (5, 2, 3, -1, 0, 4.7), (x_1^0x_2^0, x_1^1x_2^0, x_1^0x_2^1, x_1^1x_2^1, x_1^2x_2^0, x_1^0x_2^2) \rangle$$

$(5, 2, 3, -1, 0, 4.7) \longrightarrow \vec{p}$ (will drop $\rightarrow$)

$\phi_D(x)$ is the vector of monomials of $x$ up to degree $D$.

of dimension $\binom{n+D}{D}$ (here, $D = 2$)

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(will drop $\rightarrow$)

$\phi_D(x)$ is the vector of monomials of $x$ up to degree $D$.

of dimension $\binom{n+D}{D}$ (here, $D = 2$)

$$= \sum_{\substack{\alpha = (\alpha_1, \dots, \alpha_D) \\ \text{s.t. } |\alpha| \le D}} \vec{p}_\alpha \, x^\alpha \quad \text{where} \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

(here, $n = 2$)

# semidefinite programming

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$$\min_{y} \ \langle c, y \rangle \quad \text{s.t.} \quad \sum_{\gamma} y_\gamma A^i_\gamma \succeq 0 \quad \forall i, \quad Cy = b$$

linear objective

PSD cone constraints

affine constraints

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one PSD constraint

infinitely many linear constraints

$$P \succeq 0 \quad \Longleftrightarrow \quad v^T P v \geq 0 \;\; \forall v$$

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## Encoding the constraints

PSD cone is pointed:

$$P \succeq 0 \ \wedge \ -P \succeq 0 \ \Rightarrow \ P = 0$$

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## Encoding the constraints

PSD cone is pointed:

$$P \succeq 0 \wedge -P \succeq 0 \Rightarrow P = 0$$

Given $P \succeq 0$, $P \neq 0$

$$g_i(x) \geq 0 \iff g_i(x) P \succeq 0$$

$0 \preceq$

$$0 \preceq \phi_D(x)\, \phi_D(x)^T$$

$$
\begin{aligned}
0 &\preceq \phi_D(x)\,\phi_D(x)^T \\
&= \left[\phi_D(x)_\alpha\,\phi_D(x)_\beta\right]_{\alpha,\beta}
\end{aligned}
$$

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&= \left[x^\alpha\,x^\beta\right]_{\alpha,\beta} \qquad x^\alpha x^\beta = x^\gamma \quad \text{for} \quad \alpha+\beta=\gamma
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&= \sum_\gamma x^\gamma I_\gamma
\end{aligned}
$$

$$
\begin{array}{ccc}
 & & \beta \\
 & & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\alpha & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \\ & & & & & \end{bmatrix}
\end{array}
\qquad
\begin{array}{l} n = 1 \\ \gamma = 3 \end{array}
$$

$x^\alpha x^\beta = x^\gamma$ for $\alpha + \beta = \gamma$

$I_\gamma$ is matrix with 1s in these cells.

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$$\begin{array}{cc} & \beta \\ \alpha & \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & & & & 1 & & \\ 1 & & & 1 & & & \\ 2 & & 1 & & & & \\ 3 & 1 & & & & & \\ 4 & & & & & & \end{array} \end{array}$$

$n = 1$
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$$0 \preceq g_i(x)\,\phi_D(x)\,\phi_D(x)^T \iff g_i(x) \geq 0$$

$$
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0 \preceq g_i(x)\,\phi_D(x)\,\phi_D(x)^T \iff g_i(x) \geq 0
$$

$$
= \left(\sum_\alpha g^i_\alpha x^\alpha\right)\left(\sum_\beta I_\beta\, x^\beta\right)
$$

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$$= \sum_\alpha \sum_\beta g^i_\alpha I_\beta\, x^{\alpha+\beta}$$

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\alpha & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
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\hline
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\hline
\end{array}$$

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$$= \sum_\gamma x^\gamma \underbrace{\sum_{\alpha+\beta=\gamma} g^i_\alpha I_\beta}_{A^i_\gamma}$$

## The relaxation

$$\min_{x} \sum_{\gamma} p_\gamma x^\gamma \quad \text{s.t.} \quad \sum_{\gamma} x^\gamma A^i_\gamma \succeq 0 \quad \forall\, 1 \le i \le m$$

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$\vdots$

linearize $x^\alpha$ to $y_\alpha$

$\vdots$

$$\min_{y} \sum_{|\gamma| \le d} p_\gamma y_\gamma \quad \text{s.t.} \quad \sum_{|\gamma| \le D} y_\gamma A^i_\gamma \succeq 0 \quad \forall\, 1 \le i \le m$$

The relaxation

$$\min_{x} \sum_{\gamma} p_\gamma x^\gamma \quad \text{s.t.} \quad \sum_{\gamma} x^\gamma A^i_\gamma \geq 0 \quad \forall 1 \leq i \leq m$$

⋮

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$$\min_{y} \sum_{|\gamma| \leq d} p_\gamma y_\gamma \quad \text{s.t.} \quad \sum_{|\gamma| \leq D} y_\gamma A^i_\gamma \geq 0 \quad \forall 1 \leq i \leq m$$

$x^0 = 1$. Could "cheat" by taking $y_0 \to \pm\infty$ $\qquad y_0 = 1$

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linearize $x^\alpha$ to $y_\alpha$

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$$\Updownarrow$$

$$g_0(x) = 1, \; 0 \preceq g_0(x)\,\phi_D(x)\,\phi_D(x)^T$$

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# Moment relaxation of order $D$

$$v_{pop} = \min_{x} \quad p(x) \quad \text{s.t.} \quad 1 = g_0(x) \geq 0, \; g_1(x) \geq 0, \; \dots, \; g_m(x) \geq 0$$

$$v_{mom}^{D} = \min_{y} \sum_{|\gamma| \leq d} p_\gamma y_\gamma \quad \text{s.t.} \quad \sum_{|\gamma| \leq D} y_\gamma A_\gamma^i \succeq 0 \quad \forall\, 0 \leq i \leq m$$

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where $A_\gamma^i = \sum_{\alpha+\beta=\gamma} g_\alpha^i I_\beta$

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- $\binom{n+D}{D}$ variables
- Lasserre hierarchy: $v_{mom}^1 \leq v_{mom}^2 \leq \dots \leq v_{pop}$
- Under some conditions, for big enough D, it is <u>tight</u>!

$v_{mom}^{D^*} = v_{pop}$

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of appropriate size

Conversely, every $Q \succeq 0$ defines a sum of squares:

$$Q = S^T S = \begin{bmatrix} | & & | \\ s_1 & \cdots & s_k \\ | & & | \end{bmatrix} \begin{bmatrix} — & s_1 & — \\ & \vdots & \\ — & s_k & — \end{bmatrix}$$

# Abstraction

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$$\min_{y} \sum_{|\gamma| \le d} p_\gamma y_\gamma \qquad \text{s.t.} \quad y_0 = 1$$

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$$L_i y \ge 0$$

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$$\boxed{\begin{aligned} & g_i(x)\, \phi_D(x)\, \phi_D(x)^T \\ &= \Big(\sum_\alpha g^i_\alpha x^\alpha\Big)\Big(\sum_\beta I_\beta x^\beta\Big) \\ &= \sum_\alpha \sum_\beta g^i_\alpha I_\beta \cdot x^{\alpha+\beta} \\ &= \sum_\gamma x^\gamma \underbrace{\sum_{\alpha+\beta=\gamma} g^i_\alpha I_\beta}_{A^i_\gamma} \end{aligned}}$$

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$$C = \begin{bmatrix} 1 & 0 & 0 & \cdot \\ 0 & 0 & 0 & \cdot \\ 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} = C^T, \quad \text{so} \quad p - C^T l = p - \langle (1, 0, \ldots), l \rangle$$

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<u>The dual cone</u>

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## The dual cone

$$L_i y \geq 0 \quad \forall 0 \leq i \leq m$$

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$$L_i y \succeq 0 \quad \forall\, 0 \le i \le m$$

$$\equiv \ y \in L_i^{-1} Z \quad \forall\, 0 \le i \le m \qquad Z = \text{PSD cone}$$

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$$K = K_0 + \dots + K_m = \left\{ \sum_{i=0}^{m} R_i \ : \ R_i \in K_i \right\}$$

$\mathbb{R}^n$ $\qquad$ $\mathbb{R}^{\binom{n+D}{D}}$ $\qquad$ $\left(\mathbb{R}^{\binom{n+D}{D}}\right)^2$

$x$ $\qquad$ $\phi(x)$ $\qquad$ $L_i(\phi(x))$

$L_i(y)$


$L_i(K_i^*) \subseteq Z = Z^*$

$q_i : \mathbb{R}^n \to \mathbb{R}$

$q_i(x) = \langle Q_i , L_i(\phi(x)) \rangle$

For $Q_i \in Z$

$\langle Q_i, P \rangle \geq 0 \quad \forall \; P \in Z$

$\langle Q_i, P \rangle \geq 0 \quad \forall \; P \in L_i K_i^*$

$\langle Q_i, L_i y \rangle \geq 0 \quad \forall \; y \in K_i^*$

$\langle L_i^T Q , y \rangle \geq 0 \quad \forall \, y \in K_i^*$

$\to$ subset of $K_i$

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## What is $K_i$ ?

$$K_i^* = \{ y : L_i y \succeq 0 \}$$

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AKA the truncated quadratic module

to degree D

## The SDP hierarchies

$$v_{pop} = \min.\ p(x) \quad \text{s.t.} \quad g_1(x) \geq 0, \ldots, g_m(x) \geq 0$$

$$v_{mom}^D = \min_{y}.\ \langle p, y \rangle \quad \text{s.t.} \quad [y_\alpha y_\beta]_{|\alpha|, |\beta| \leq D} \succeq 0$$

$$\sum_{|\gamma| \leq D} y_\gamma A_\gamma^i \succeq 0 \quad \forall 1 \leq i \leq m$$

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$$v_{sos}^D = \max_{l_0}\ l_0 \quad \text{s.t.} \quad p - l_0 = q_0 + \sum_{i=1}^{m} g_i q_i$$

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by weak duality.
(Strong duality under usual conditions)

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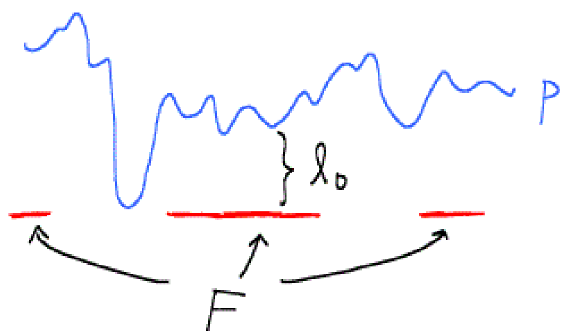


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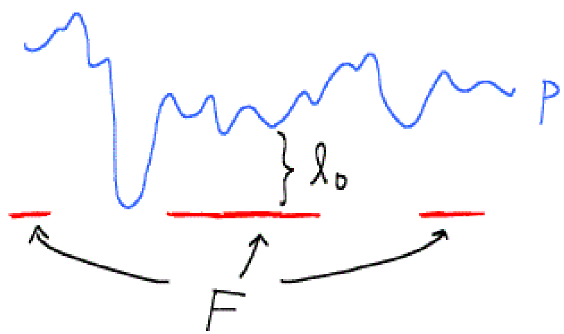


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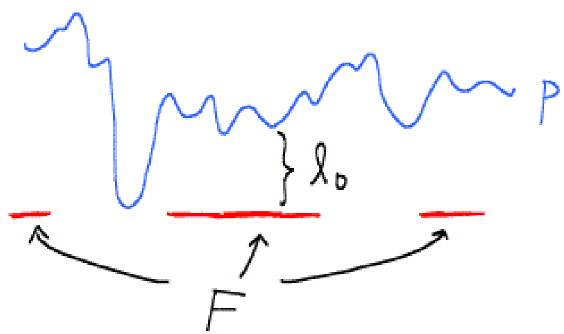


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$\Rightarrow$ $p - l_0$ has a certificate of nonnegativity (at round $D$)

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This is Putinar's Positivstellensatz, a specialization of a central result in algebraic geometry.

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  $p$ has a unique minimum $\Rightarrow$ $x_i = x^{e_i} = y_{e_i}$
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- Sparsity

Go forth and optimize!