CONIC DUALITY

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The key concept in this recitation:

$\boxed{\text{A vector } v \text{ defines a valid inequality for a set } S \text{ if: } \forall x \in S,\ \langle v, x \rangle \geq 0}$

That 'for all' is ultimately why we can use duality to certify properties like lower bounds.

Agenda:

- Basic cone definitions
- Cone-induced inequalities: use to generalize linear programs to cone programs
- The conic dual program and the dual cone
- Symmetry between the primal and dual conic programs
- Combinatorial dual cones
- Easily move between primal V representation and dual H-representation

Cone: closed under nonnegative scaling

We will restrict attention to proper cones.

convex, pointed, closed, nonempty, closed under +

halfspace representation

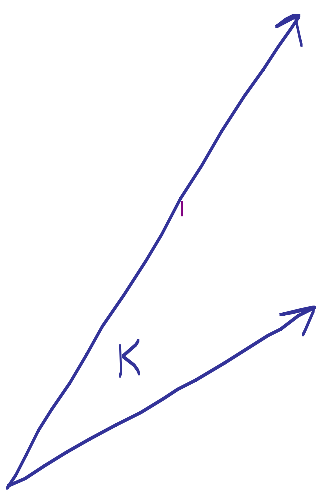
$$K_H(A) = \{ x : Ax \geq 0 \}$$

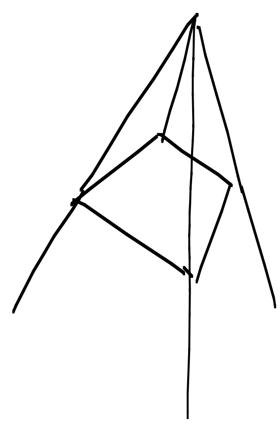
actually, ray

vertex representation

$$K_V(R) = \{ R\lambda : \lambda \geq 0 \}$$

polyhedral if A or R is finite





lift of polytope

e.g. nonneg. orthant, SOC, PSD, copositive

Generalized inequality:

$a \geq_K b$ if $a - b \in K$

usual vector inequality induced by nonneg. orthant.

$\min \quad \langle c, x \rangle \quad \text{s.t.} \quad Ax \geq b \qquad (LP)$

$\min \quad \langle c, x \rangle \quad \text{s.t.} \quad Ax \geq_K b \qquad (P)$

duality: combining given inequalities to produce a new one. How?

for LP: $Ax \geq b \Rightarrow$

$$\lambda_1 A_1 x + \dots + \lambda_m A_m x \geq \lambda_1 b_1 + \dots + \lambda_m b_m$$

$$\langle c, x \rangle \overset{\text{enforce}}{=\!=} \langle A^T \lambda, x \rangle \overset{\text{adjoint def}}{=} \langle \lambda, Ax \rangle \geq \langle \lambda, b \rangle$$

$\max \quad \langle b, \lambda \rangle \quad \text{s.t.} \quad A^T \lambda = c$

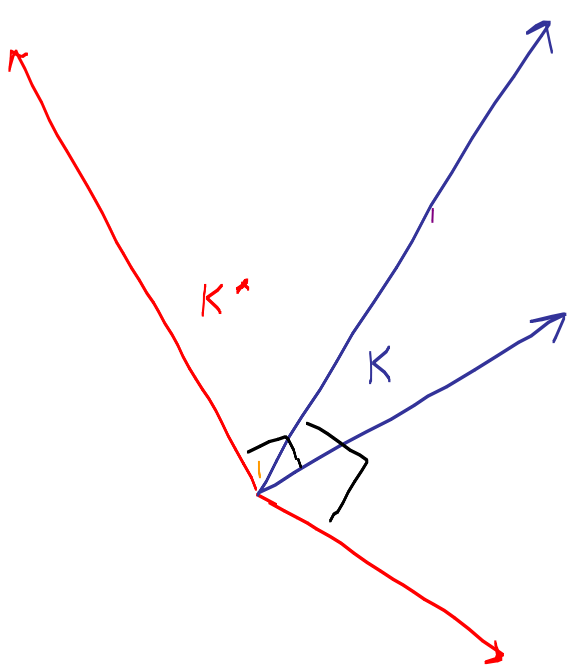
works for $\lambda \geq 0$

for CP: $\{ \lambda : Ax \geq_K b \Rightarrow \langle \lambda, Ax \rangle \geq \langle \lambda, b \rangle \}$

with $a = Ax - b$

$$\{ \lambda : a \geq_K 0 \Rightarrow \langle \lambda, a \rangle \geq 0 \}$$

$$\{ \lambda : \forall a \in K, \quad \langle \lambda, a \rangle \geq 0 \} = K^*$$



The dual cone is just the vectors that define valid inequalities for K.

now swap $\lambda \geq 0$ for $\lambda \in K^*$ in LP dual for CP dual

$\max \quad \langle b, \lambda \rangle \quad \text{s.t.} \quad A^T \lambda = c, \quad \lambda \in K^* \qquad (D)$

$\lambda \in L^*$

Let's make primal also be linear objective under affine equality and cone membership constraints.

$$\min \langle c, x\rangle \quad \text{s.t.} \quad \underbrace{Ax - b}_{} \in K$$

want linear objective in this, i.e. d s.t.

$\langle c, x\rangle = \langle d, Ax-b\rangle$

up to constant in x

$\langle c, x\rangle + \text{const}$

$= \langle d, Ax-b\rangle = \langle d, Ax\rangle - \langle d, b\rangle = \langle A^T d, x\rangle + \text{const}$

Thus $c = A^T d$ so d exists when $c \in Im(A^T)$

e.g. $\text{rank}(A) = n$: $d = (A^T)^{-1} c = (A^{-1})^T c$

So primal rewritten is

$$\min \langle d, y\rangle \quad \text{s.t.} \quad y = Ax - b, \quad y \in K$$

for some x

$y \in L$

$= Im(A) - b$

where $c = A^T d$

$L = \{ Ax - b : x \in \mathbb{R}^n \}$

$= \mathcal{L} - b$ where

$\mathcal{L} = \{ Ax : x \in \mathbb{R}^n \}$

$L^* = \{ \lambda \in \mathbb{R}^m : A^T \lambda = c \}$

$= \mathcal{L}^* + d$ where

$\mathcal{L}^* = \{ \lambda \in \mathbb{R}^m : A^T \lambda = 0 \}$

$= \{ \lambda \in \mathbb{R}^m : A^T(\lambda + d) = c \}$

$A^T\lambda + A^T d = c$

Image of A

Null space of A^T

Those subspaces are orthogonal complements. $(\mathcal{L}^* = \mathcal{L}^\perp)$

$$A^T\lambda = \begin{bmatrix} b_1 \cdot \lambda = 0 \\ \vdots \\ b_n \cdot \lambda = 0 \end{bmatrix}$$

Finally, nicely geometric / symmetric reformulation:

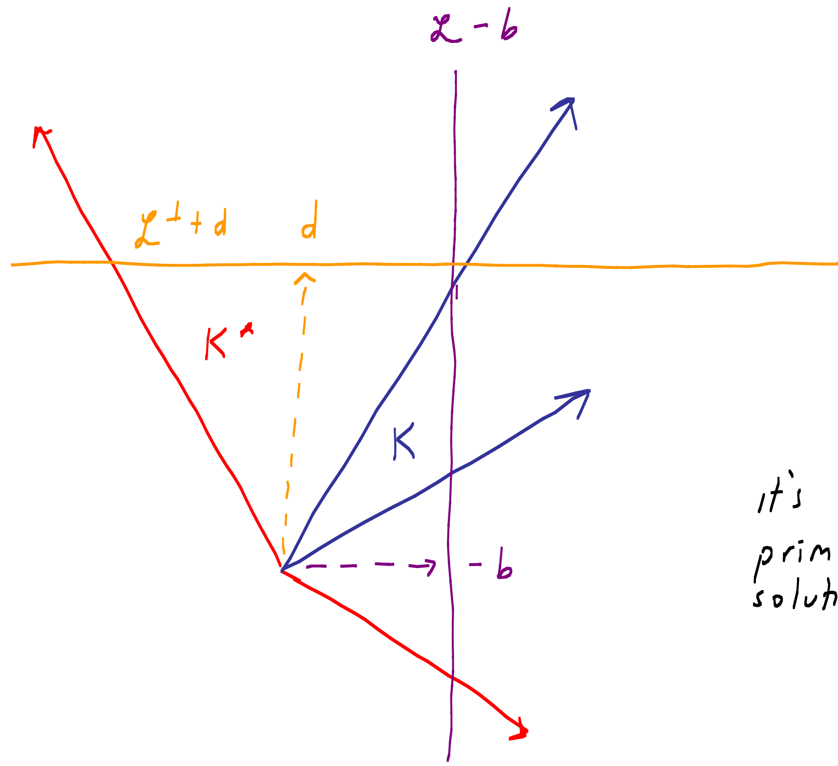
$$\min \ \langle d, y \rangle \quad \text{s.t.} \quad y \in \mathcal{L} - b, \ y \in K \qquad (P)$$

$$\max \ \langle b, \lambda \rangle \quad \text{s.t.} \quad y \in \mathcal{L}^\perp + d, \ y \in K^* \qquad (D)$$

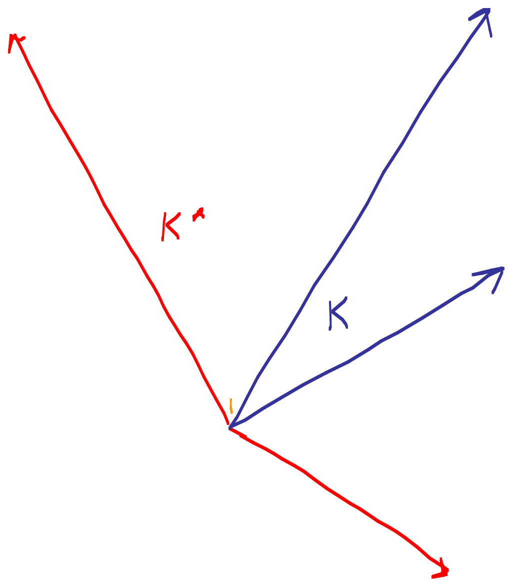
$\mathcal{L} = \mathrm{Im}(A)$

$c = A^T d$

$\mathcal{L}^{\perp\perp} = \mathcal{L}, \quad K^{**} = K$



it's easy to find the primal and dual solutions on the diagram.



vertices   edges

Rays of primal are halfspaces of dual? Indeed:

$$K_H(A)^* = K_V(A^T)$$

In general, this interchange corresponds to a combinatorial definition of duality.

If a valid v's induced hyperplane

$$H_v = \{x : \langle v, x \rangle = 0\}$$

has a non-empty intersection with K, then H_v is said to be a supporting hyperplane, and the intersection is called a non-empty face of K.

- 0 is in every face of K
- Each face associated with active rays

Partial order of faces in terms of inclusion

Combinatorial dual: 1-to-1 map between respective faces of primal and dual which reverses partial order

<u>Thm</u>: $K_H(A)$ and $K_V(A^T)$ are combinatorial duals.

Proof: exhibit mapping between proper faces.
(Non-proper are trivial.)

F is a nonempty face of $K_H(A)$

$$= \{ x \in C_H(A) \; : \; \underbrace{A_I}_{\text{active inequalities at } F} x = 0 \}$$

$\exists \, v$ s.t. $\forall i \in I \quad A_i v = 0 \quad \leftrightarrow \quad v^T (A_i)^T = 0$

$\forall j \notin I \quad A_j v > 0 \quad \leftrightarrow \quad v^T (A_j)^T > 0$

v defines a valid inequality for $K_V(A^T)$:

$\forall x \in K_V(A^T)$, $\qquad x = A^T \lambda$

$$v^T x = v^T(\lambda_1 (A_1)^T + \dots + \lambda_m (A_m)^T) \qquad \lambda \geq 0$$

$$= \lambda_1 \underbrace{v^T (A_1)^T}_{\geq 0} + \dots + \lambda_m \underbrace{v^T (A_m)^T}_{\geq 0}$$

Claim:

$\underbrace{H_v \cap K_V(A^T)}_{F'}$ is a face. $(A_I)^T$ are active generators

Need:

$H_v \cap K_V(A^T)$ is non-empty.

To show:

start with $x \in F'$ (can always let $x = 0$), then show that

$$x + \theta (A_I)^T \in \underset{①}{H_v}, \underset{②}{K_v}(A^T) \qquad \theta \geq 0$$

① $$v^T(x + \theta A_I^T) = \underset{0}{v^T x} + \theta \underset{0}{(A_I v)^T} = 0$$

② $$\begin{aligned} x &= \lambda_1 (A_1)^T + \cdots\cdots + \lambda_n (A\ )^T \\ x + \theta (A_I)^T &= {}'' + \cdots + \underbrace{(\lambda_i + \theta)}_{\text{still nonnegative}} (A_i)^T + \lambda\ (A\ )^T \\ &\in K_v(A^T) \end{aligned}$$

Note that if we started out with a lot of active inequalities (i.e. a low-dimensional face), we get a face with a lot of active generators (i.e. a high-dimensional one.)

References

Lectures on Modern Convex Optimization. Ben Tal and Nemirovski.

Polyhedral Computation, Spring 2011. Fukuda